

Part I. No calculators are allowed. Please do not show your work for Part I. Just write down the final answer on the white paper provided. Each problem is worth 5 points.

1. Compute $\sin^2 0^\circ + \sin^2 1^\circ + \cdots + \sin^2 89^\circ + \sin^2 90^\circ$.

Solution. Note that $\sin x^\circ = \cos(90 - x)^\circ$, and since $\sin^2 \theta + \cos^2 \theta = 1$ for any θ , we have

$$\begin{aligned} \sin^2 0^\circ + \sin^2 1^\circ + \cdots + \sin^2 89^\circ + \sin^2 90^\circ &= \sin^2 0^\circ + \sin^2 1^\circ + \cdots + \sin^2 44^\circ + \sin^2 45^\circ \\ &\quad + \cos^2 44^\circ + \cdots + \cos^2 1^\circ + \cos^2 0^\circ \\ &= 1 + 1 + \cdots + 1 + \sin^2 45^\circ \\ &= \frac{91}{2} \end{aligned}$$

2. Rectangle $ABCD$ has sides AB and BC in ratio $3 : 1$. If the diagonal AC has length 5, find the area of the rectangle.

Solution. If BC has length x , then AB has length $3x$ and by the Pythagorean Theorem, we have $10x^2 = 25$. It follows that the area of the rectangle is $3x^2 = 15/2$.

3. If x, y, z are positive real numbers such that $xyz = 1$, $x + 1/z = 5$, and $y + 1/x = 29$, find $z + 1/y$.

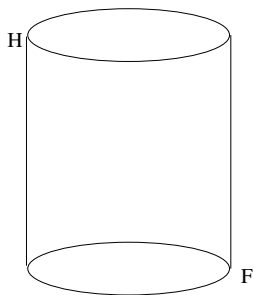
Solution. Using $1/z = xy$, we have $x(1 + y) = 5$ and then putting this into the third equation, we discover that $y = 24$. Then $x = 1/5$ and $z = 5/24$ and therefore $z + 1/y = 1/4$.

4. The polynomial $p(x) = x^5 - 3x^4 - 11x^3 + 27x^2 + 10x - 24$ has five real roots. Find the sum of the squares of the roots.

Solution. Call the roots a, b, c, d, e . Note that $a + b + c + d + e = 3$ and $ab + ac + ad + ae + bc + bd + be + cd + ce + de = -11$. Then $(a + b + c + d + e)^2 = 9$ and

$$a^2 + b^2 + c^2 + d^2 + e^2 = (a + b + c + d + e)^2 - 2(ab + ac + ad + ae + bc + bd + be + cd + ce + de) = 31.$$

5. A cylindrical cup is four inches high and six inches in circumference. On the top edge of the cup is a drop of honey. Directly opposite, on the bottom edge of the cup, is a fly. What is the minimum distance the fly must walk to reach the honey?



Solution. Cut open the cup and lay it out flat. The fly can walk along the hypotenuse of the right triangle joining the two points. This triangle has side lengths 3 and 4 and so the distance the fly must walk is 5 inches.

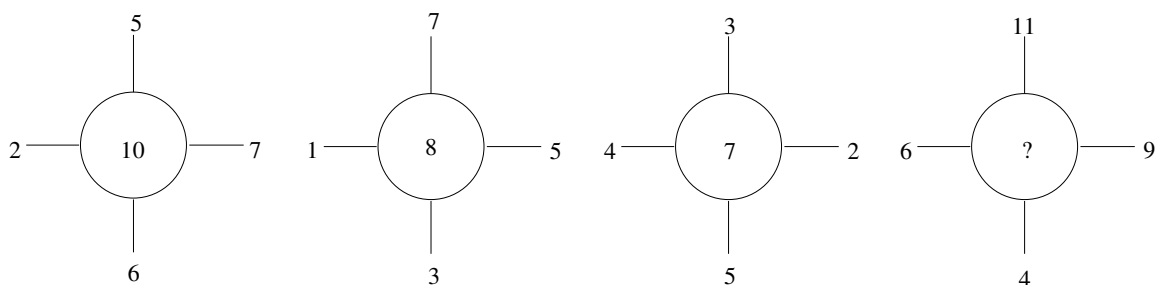
6. There are a number of planes on an island. The fuel tank of each plane, when filled, contains enough fuel to take it halfway around the world. It is possible to transfer fuel from one plane to another during flight, but there is no other source of fuel available. The pilots are told that one plane must fly around the world and that every plane must return to the island. What is the absolute minimum number of planes needed so that one of them can fly around the world?

Solution. The minimum number of planes is three. Call the planes A, B & C. They start off together with full tanks of fuel. When they get $1/8$ of the way around the world they have used $1/4$ of their fuel. C then divides his fuel into three equal parts; having $3/4$ of its fuel left, it transfers $1/4$ to A, $1/4$ to B, and uses the remaining $1/4$ to fly back to the island. A and B now have full tanks again. They fly on till they reach $1/4$ of the way around. Both then have $3/4$ of their fuel left. B then transfers $1/4$ of its fuel to A, because the remaining $1/2$ is needed to return to the base. A now has a full tank of fuel and uses it to get $3/4$ of the way around where he is met by C who has refueled and flown from the island the other way, using the fact that the earth is round. To reach the meeting point, C has used up $1/2$ of its fuel. He gives half the remainder to A, so that they both reach $7/8$ of the way, where they are met by B. To reach this point, B has used up $1/4$ of its fuel. It needs another $1/4$ to return to the island, but divides the remainder between A and C. Now A, B, and C all return to the island, A having successfully flown around the earth.

7. All of the residents of a certain island are either mathematicians or politicians. Everything the mathematicians say is the truth, and everything the politicians say is a lie. One day you run into Alex, Bob, and Carl. Alex says: "All three of us are politicians." Bob says: "Exactly one of us is a mathematician." Determine whether each of Alex, Bob, and Carl are mathematicians or politicians.

Solution. Note that Alex cannot be a mathematician, for if he were, his statement would be false, and mathematicians never lie. So Alex is a politician. Now, Bob cannot be a politician, for if he were his statement would be false so that there would be at least two mathematicians. So Bob is a mathematician, and therefore Carl must be a politician since there is exactly one mathematician.

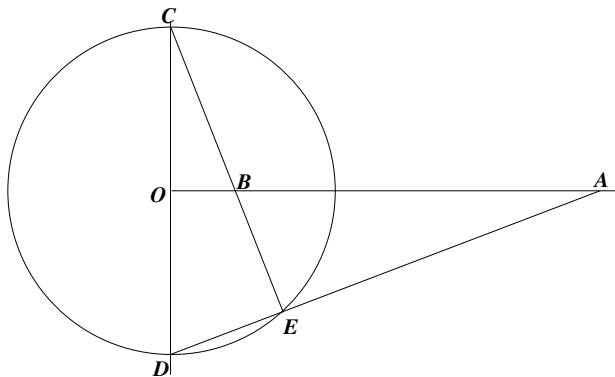
8. For each of the four wheels, the numbers around the wheel and the number in the middle of the circle bear a certain relationship to each other. (The same relationship for all four wheels.) What is the missing number?



Solution. In each case, the number in the center is half the sum of the four numbers around it. The missing number is therefore 15.

Part II. Please show all work on the yellow paper provided. Each problem is worth 20 points.

9. In the figure below, the circle centered at O has radius 1 and segment CD is perpendicular to segment OA . If the segment OA has length r , find the length of segment OB .



Solution. I claim that triangle AOD is similar to triangle COB . To see this, note that since angle ABE equals angle CBO , and since triangle COB is a right triangle, it suffices to show that segment CE is perpendicular to segment AD . For simplicity, denote angle ECD by α and angle CDE by β . Then the lengths of the arcs swept out by these angles are 2α and 2β , respectively. Thus, $2\alpha + 2\beta = \pi$; that is, $\alpha + \beta = \pi/2$. But then angle CED is also $\pi/2$ and so segment CE is perpendicular to segment AD .

Now, given this, we have the equation

$$\frac{OC}{OB} = \frac{OA}{OD},$$

which tells us that OB has length $1/r$.

10. A *palindrome number* is an integer that reads the same backwards as forwards (e.g., 2468642). Let X be the set of all 11-digit positive numbers. An integer is chosen at random from X . Compute the probability that it is a palindrome number.

Solution. There are 9×10^{10} numbers in X , and of these, 9×10^5 are palindromes. The probability is therefore $1/10^5$.

11. If eggs are removed from a basket 2, 3, 4, 5, and 6 at a time, there remain, respectively, 1, 2, 3, 4, and 5 eggs. But if the eggs are removed 7 at a time, no eggs remain. What is the least number of eggs that could have been in the basket?

Solution. There were at least 119 eggs in the basket. This is seen by solving the following system of congruences.

$$X \equiv 1 \pmod{2}$$

$$X \equiv 2 \pmod{3}$$

$$X \equiv 3 \pmod{4}$$

$$X \equiv 4 \pmod{5}$$

$$X \equiv 5 \pmod{6}$$

$$X \equiv 0 \pmod{7}$$