**Discovery Day Mathematics Exam** **February 17, 2006**

**Part I**. No calculators are allowed. Please do not show your work for Part I. Just write down the final answer on the white paper provided. Each problem is worth 5 points.

1. The triple $(a, b, c)$ of real numbers has the property that $a+b+c = 5$ and $ab+ac+bc = 5$. What are the possible values of $a^2 + b^2 + c^2$?

**Solution.** We have the following:

$$\begin{aligned} 25 &= (a + b + c)^2 \\ &= (a^2 + b^2 + c^2) + 2(ab + ac + bc) \\ &= (a^2 + b^2 + c^2) + 10. \end{aligned}$$

It follows that $a^2 + b^2 + c^2 = 15$.

2. A rectangular sheet of paper $ABCD$ is three inches in width and four inches in length. The paper is folded so that the two diagonally opposite corners $A$ and $C$ coincide. Determine the length of the crease in the paper.

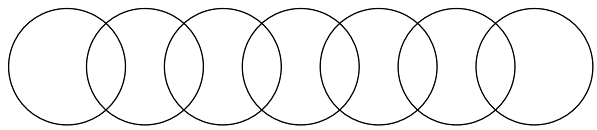
**Solution.** The crease makes a line segment cutting across the rectangle; call it $FG$ and label the ends so that the points are in the following order around the rectangle: $ABGCDF$. Assume that $AB$ has length 3. Then $AC$ has length 5, and $AC$ meets $FG$ at a point $E$. Note that $AC$ and $FG$ are perpendicular bisectors of each other so that $EC$ has length 5/2. Now, triangle $ABC$ is similar to triangle $EGC$ and so we have $\overline{EG}/\overline{EC} = \overline{AB}/\overline{BC}$; that is, $\overline{EG}/(5/2) = 3/4$ and hence that $EG$ has length 15/8. The crease then has length 15/4.

3. What is the sum of the coefficients of the polynomial obtained by expanding:

$$p(x) = (1 - 3x + 3x^2)^{2005}(1 + 2x - 2x^2)^{2006}?$$

**Solution.** The sum of the coefficients of any polynomial $q(x)$ is simply $q(1)$. So the answer in this case is just $p(1) = 1$.

4. A traveller has a straight gold chain with 7 links. She plans to stay at an inn which is willing to accept one link as payment for one night's lodging. She doesn't know when she'll want to leave, so she wants to pay on a daily basis. She'll have to cut some of the links in her chain of course; the inn is perfectly willing to accept cut links as payment. What is the smallest number of links she is forced to cut in order that she is able to pay every night (possibly getting change in the form of links paid earlier) for up to a week? Which link(s) should she break?



**Solution.** She should break just one, the third link. If she does this, she can pay for the first night with it, leaving her with two chains of length 2 and 4. The second night, she pays with the chain of length 2, receiving one length as change, so that she then has a chain of length 4 and one of length 1, while the inn has a single chain of length 2. I'll leave it to you to figure out how she should pay on each subsequent night.

5. Given that $13 = 2^2 + 3^2$ and $74 = 5^2 + 7^2$, express $13 \times 74 = 962$ as a sum of two squares.

**Solution.** The best thing to do here is to use complex numbers. Let $z = 2 + 3i$ and $w = 5 + 7i$. Then use the fact that $|zw| = |z||w|$. In this case we have $zw = -11 + 29i$, $|z| = 2^2 + 3^2 = 13$, and $|w| = 5^2 + 7^2 = 74$. Thus,

$$962 = |z||w| = |zw| = 11^2 + 29^2.$$

6. In a certain college of under 5000 total enrollment, a third of the students were freshmen, two-sevenths were sophomores, a fifth were juniors and the rest seniors. The math department offered a popular course in which were registered a fortieth of all the freshmen at the college, a sixteenth of all the sophomores, and a ninth of all the juniors, while the remaining third of the math class were all seniors. How many students were there in the math class?

**Solution.** Let $X$ denote the number of students at the college. Note that $X$ is divisible by 7, 5, 8, and 9. Since these numbers are relatively prime, $X$ is divisible by their product, which is 2520. Since $X < 5000$, we must have $X = 2520$. One then quickly discovers that there are 183 students in the class.

7. If $z$ is a real number between $\pi$ and $2\pi$, and $\sin z = -2/3$, what is $81(\sin 2z)^2$?

**Solution.** Use the identity $\sin 2\theta = 2 \sin\theta \cos\theta$ to get the following:

$$81(\sin 2z)^2 = 81 \cdot 4 \sin^2 z \cos^2 z = (81)(4)(4/9)(5/9) = 80.$$

8. Compute $\displaystyle\sum_{i=1}^{99} \frac{200}{i(i+1)}$.

**Solution.** Use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

to get the following:

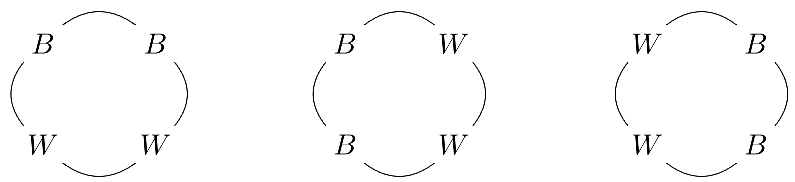
$$\begin{aligned}
\sum_{i=1}^{99} \frac{200}{i(i+1)} &= 200\left(\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)\right) \\
&= 200\left(1-\frac{1}{100}\right) \\
&= 200\cdot\frac{99}{100} \\
&= 198.
\end{aligned}$$

**Part II**. Please show all work on the yellow paper provided. Each problem is worth 20 points.

9. Let $n > 1$ be a fixed positive integer. There are $n$ coins on a table with each showing tails. A move consists of turning over all but one coin. For which values of $n$ can one have all coins showing heads after a certain number of moves?

10. A villager pays rent to a local farmer who allows him to graze his pet goat on his (the farmer's) field. The goat is tethered to a post by a 21 foot rope, so it grazes in a 21 foot circle. The farmer has decided to build a 14 foot by 7 foot shed in the field using the post as one of the corners of the shed. If the farmer has been charging the villager \$100 per year, by how much should the rent be reduced to reflect the lost grazing area?

11. A group of children has decided to make some necklaces. They have an unlimited supply of round beads in two colors: black and white. Each necklace will have exactly four beads on it, and any combination of colors may be used (that is, repeating colors is allowed). How many different necklaces can the children make? For example, note that the following three necklaces are all the same.




The second is obtained from the first by counterclockwise rotation, while the third is obtained from the second by flipping it over. We would not call these different necklaces, right?