Some geometric aspects of the fundamental group at infinity

Hanspeter Fischer

Department of Mathematical Sciences
Ball State University

Abstract

It is a longstanding, and still unresolved, conjecture that Euclidean 3-space is the only open contractible 3-dimensional manifold, which can cover a compact manifold. In all dimensions 4 and higher, we know this to be false, because of Davis' exotic examples. What we do have in dimension 3, is the following theorem by Hass, Rubinstein, and Scott: Every closed aspherical (irreducible) 3-manifold whose fundamental group contains the fundamental group of a closed aspherical surface, is covered by Euclidean space. The large scale topology in all of the above is controlled by the fundamental group at infinity. For example, an (irreducible) open contractible manifold M, of any dimension $n > 2$, will have trivial fundamental group at infinity if and only if it is homeomorphic to Euclidean space. What makes Davis' examples so interesting is their very rich topology at infinity. We will discuss methods for analyzing the degree of complexity of the topology at infinity of Davis' examples. We will also present conditions which prevent such bad behavior at infinity from occurring altogether, thus obtaining higher dimensional variations of the Hass-Rubinstein-Scott result. Finally, we will turn to another setting in which the fundamental group at infinity is of interest, namely that of non-positively curved geodesic spaces. Such spaces come naturally equipped with a visual boundary at infinity: the set of infinite geodesic rays emanating from a fixed base point, endowed with the compact-open topology. There is an intriguing connection between the fundamental group of such a visual boundary and the fundamental group at infinity of the underlying space. We will discuss scenarios in which the former group naturally embeds into the latter. These scenarios include, but are not limited to, the Davis-style examples.